

# Composition Kostka functions

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Macdonald defined two-parameter Kostka functions  $K_{\lambda\mu}(q, t)$  where  $\lambda, \mu$  are partitions. The main purpose of this paper is to extend his definition to include all compositions as indices. Following Macdonald, we conjecture that also these more general Kostka functions are polynomials in  $q$  and  $t^{1/2}$  with non-negative integers as coefficients. If  $q = 0$ , then our Kostka functions are Kazhdan-Lusztig polynomials of a special type. Therefore, our positivity conjecture combines Macdonald positivity and Kazhdan-Lusztig positivity and hints towards a connection between Macdonald and Kazhdan-Lusztig theory.

## 1. Introduction

In [M1], Macdonald introduced a new class of symmetric functions  $J_\mu(z; q, t)$ , parameterized by a partition  $\mu$  and depending on two parameters  $q$  and  $t$ , which generalizes both Hall-Littlewood and Jack polynomials. In the same paper, he introduced the two variable Kostka functions

$$(1.1) \quad K_{\lambda\mu}(q, t) := \langle s_\lambda, J_\mu \rangle_{\text{HL}}$$

where  $s_\lambda$  is the Schur functions for the partition  $\lambda$  and where  $\langle \cdot, \cdot \rangle_{\text{HL}}$  denotes the scalar product rendering the Hall-Littlewood polynomials orthogonal. Based on computational evidence and some special cases, Macdonald conjectured that  $K_{\lambda\mu}(q, t)$  is a polynomial in  $q, t$  with non-negative integral coefficients. Even polynomiality was open for a while and was almost simultaneously proved in [GR], [GT], [Ki], [Kn1], [Sa]. Haiman finally proved positivity in [Ha].

To prove polynomiality, the author used in [Kn1] a more general theory, that of non-symmetric Macdonald polynomials which has been developed mainly by Cherednik. For that reason, it is tempting to look for Kostka functions associated to non-symmetric Macdonald polynomials and prove their positivity first. In this paper, we introduce functions  $K_{\lambda\mu}(q, t)$  where  $\lambda$  and  $\mu$  are now allowed to be compositions, i.e., finite unordered sequences

of positive integers and which coincide with Macdonald's when  $\lambda$  and  $\mu$  are partitions. This definition links two theories, Macdonald and Kazhdan-Lusztig, which, even though they share the same background, namely affine Hecke algebras, have been unrelated so far.

The starting point of our theory was Lusztig's observation, [Lu1], [Lu2], that certain Kazhdan-Lusztig basis elements can be identified with the Schur function. Therefore, the idea is roughly to replace in (1.1) the symmetric Macdonald polynomial by a non-symmetric one and the Schur function  $s_\lambda$  by a Kazhdan-Lusztig basis element.

More precisely, we consider the standard parabolic module  $\mathcal{M}$  of the (extended) affine Hecke algebra of type  $A_{n-1}$ . This module can be identified with a polynomial ring and has a basis consisting of the non-symmetric Macdonald polynomials  $\mathcal{E}_\mu$ . On the other hand,  $\mathcal{M}$  has also the canonical, or Kazhdan-Lusztig basis  $\underline{M}^\lambda$ . This basis is constructed from the standard basis  $M^\lambda$  by forcing selfduality. Moreover,  $\mathcal{M}$  carries the scalar product on  $\mathcal{M}$  for which the standard basis is orthonormal. Then we define  $K_{\lambda\mu}^{(n)} := \langle \underline{M}^\lambda, \mathcal{E}_\mu \rangle$ .

So far, the construction works more or less for any root system but sample calculations show that  $K_{\lambda\mu}^{(n)}$  does not have positive coefficients, even in type  $A_{n-1}$ . For this to happen, we have to stabilize, i.e., let the number  $n$  tend to  $\infty$ . If we equip  $\mathbb{Z}[q, t^{1/2}, t^{-1/2}]$  with the  $t$ -adic topology, then we show that  $K_{\lambda\mu} = \lim_{n \rightarrow \infty} K_{\lambda\mu}^{(n)}$  exists and is an element of  $\mathbb{Z}[q, t^{1/2}, t^{-1/2}]$ . This is the main (proven) result of this paper.

We conjecture that  $K_{\lambda\mu}(q, t)$  has positive coefficients. This has been confirmed in a great number of cases by direct computation. Further evidence is the fact that in case  $\lambda$  and  $\mu$  are partitions then our  $K_{\lambda\mu}$  coincides with Macdonald's (proved to be positive by Haiman). This is a consequence of the aforementioned theorem of Lusztig.

Finally, we show that for  $q = 0$  the Macdonald polynomials specialize to the standard basis of  $\mathcal{M}$  (see (11.10) and also [Kn1] Cor. 5.7). This means that  $K_{\lambda\mu}(0, t)$  is a Kazhdan-Lusztig polynomial, hence positive.

At the end of the paper we present a conjecture which substantially refines Macdonald's positivity. More precisely, we define "marked" Kostka polynomials  $K_{\lambda\mu}(t)$  which do not depend on  $q$  anymore. Here  $\mu$  is a composition with some boxes of its Young diagram are marked. The unmarked polynomials  $K_{\lambda\mu}$  can be obtained in an easy and positive way from the marked ones. Ample numerical evidence suggests that the  $K_{\lambda\mu}(t)$  have positive coefficients.

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## 2. The affine Hecke algebra for $GL_n$

Consider the lattice  $X := \mathbb{Z}^n$  with standard basis  $e_1, \dots, e_n$ . Then the symmetric group on  $n$  letters,  $W_f := S_n$ , acts on  $X$  by permuting the  $e_i$ :

$$(2.1) \quad \pi(\tau_1, \dots, \tau_n) := (\tau_{\pi^{-1}(1)}, \dots, \tau_{\pi^{-1}(n)}).$$

Let  $W := W_f \ltimes X$  be the (extended) affine Weyl group. For  $\tau \in X$  let  $t_\tau$  be the corresponding element in  $W$ .

Let  $X^\vee := \text{Hom}(X, \mathbb{Z}) \cong \mathbb{Z}^n$  with basis  $\varepsilon_1, \dots, \varepsilon_n$  (the basis dual to  $e_1, \dots, e_n$ ). Let  $\Delta_f \subseteq X^\vee$  be the set of roots for  $W_f$ , i.e.,

$$(2.2) \quad \Delta_f := \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq n\}$$

We regard  $a \in \mathbb{Z}$  as the constant function  $a$  on  $X$ . Then the set of affine roots  $\Delta := \Delta_f + \mathbb{Z}$  consists of affine linear functions on  $X$ . Let

$$(2.3) \quad \alpha_0 := \varepsilon_n - \varepsilon_1 + 1, \alpha_1 := \varepsilon_1 - \varepsilon_2, \dots, \alpha_{n-1} := \varepsilon_{n-1} - \varepsilon_n.$$

Then  $\Sigma_f := \{\alpha_1, \dots, \alpha_{n-1}\}$  is the set of simple roots of  $\Delta_f$  while  $\Sigma := \{\alpha_0, \dots, \alpha_{n-1}\}$  is the one for  $\Delta$ . The corresponding simple reflections are denoted by  $s_i$ . Thus, for  $1 \leq i < n$  we have  $s_i = (i \ i+1)$  while  $s_0(\tau_1, \dots, \tau_n) = (\tau_n + 1, \tau_2, \dots, \tau_{n-1}, \tau_1 - 1)$ . The simple roots generate the positive roots

$$(2.4) \quad \Delta_f^+ := \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\} \subseteq \Delta_f, \quad \Delta^+ := \Delta_f^+ \cup (\Delta_f + \mathbb{Z}_{>0}) \subseteq \Delta.$$

The dominant Weyl chamber is  $X_+ := \{\tau \in X \mid \tau_1 \geq \dots \geq \tau_n\}$ .

Every element  $w \in W$  acts on  $\Delta$  by  $w\alpha(\tau) = \alpha(w^{-1}\tau)$ . We define its length as

$$(2.5) \quad \ell(w) := \#\{\alpha \in \Delta^+ \mid w\alpha \in -\Delta^+\}.$$

For  $w \in W_f$  and  $\tau \in X$  we have the useful formula

$$(2.6) \quad \ell(t_\tau w) = \sum_{\substack{1 \leq i < j \leq n \\ w^{-1}(i) < w^{-1}(j)}} |\tau_i - \tau_j| + \sum_{\substack{1 \leq i < j \leq n \\ w^{-1}(i) > w^{-1}(j)}} |\tau_i - \tau_j - 1|$$

This means, in particular, that

$$(2.7) \quad \omega := t_{-e_n} s_{n-1} \dots s_1 = s_{n-1} \dots s_1 t_{-e_1},$$

acting on  $X$  like

$$(2.8) \quad \omega(\tau_1, \dots, \tau_n) = (\tau_2, \dots, \tau_n, \tau_1 - 1),$$

has length zero. In fact,  $\omega$  generates  $\Omega := \{w \in W \mid \ell(w) = 0\} \cong \mathbb{Z}$ . If  $W^a \subseteq W$  is the subgroup generated by  $s_1, \dots, s_n$ , then  $W^a = S_n \ltimes Q$  with  $Q := \{\tau \in X \mid \sum_i \tau_i = 0\}$  and  $W = \Omega \ltimes W^a$ . The action of  $\Omega$  on  $W^a$  is given by

$$(2.9) \quad \omega s_i \omega^{-1} = s_{i-1}, \quad i = 1, \dots, n-1, \quad \omega s_0 \omega^{-1} = s_{n-1}.$$

Let  $\mathcal{L} := \mathbb{Z}[v, v^{-1}]$ . We often use also the notation  $t := v^2$ . The (extended) affine Hecke algebra  $\mathcal{H}$  is the  $\mathcal{L}$ -algebra generated by elements  $H_0, \dots, H_{n-1}, \omega$  with relations

$$(2.10) \quad H_i H_j = H_j H_i \quad \text{for } 0 \leq i < j \leq n-1 \text{ with } 1 < |i-j| < n-1$$

$$(2.11) \quad H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1} \quad \text{for } i = 0, \dots, n-1 \quad \text{with } H_n := H_0$$

and

$$(2.12) \quad \omega H_i \omega^{-1} = H_{i-1} \quad i = 1, \dots, n-1, \quad \omega H_0 \omega^{-1} = H_{n-1}$$

$$(2.13) \quad (H_i + v)(H_i - v^{-1}) = 0 \quad i = 0, \dots, n-1.$$

If  $w = s_{i_1} \dots s_{i_r} \omega^k \in W$  is a reduced expression, then one puts  $H_w := H_{i_1} \dots H_{i_r} \omega^k$ . These elements form an  $\mathcal{L}$ -basis of  $\mathcal{H}$ . Moreover

$$(2.14) \quad H_x H_y = H_{xy} \quad \text{whenever } \ell(x) + \ell(y) = \ell(xy)$$

The subalgebra spanned by  $H_w, w \in W_f$  is denoted by  $\mathcal{H}_f$ .

### 3. The parabolic module

There is a unique  $\mathcal{L}$ -linear homomorphism  $\mathcal{H}_f \rightarrow \mathcal{L}$  with  $H_1, \dots, H_{n-1} \mapsto v^{-1}$ . More generally,  $H_w \mapsto v^{-\ell(w)}, w \in W_f$ . This way,  $\mathcal{L}$  becomes a  $\mathcal{H}_f$ -module denoted by  $\mathcal{L}(v^{-1})$ . Consider the induced module

$$(3.1) \quad \mathcal{M} := \mathcal{H} \otimes_{\mathcal{H}_f} \mathcal{L}(v^{-1}).$$

Every coset in  $W/W_f$  is represented by a unique element  $t_\tau, \tau \in X$ . This implies that the elements  $H_{t_\tau} \otimes 1 \in \mathcal{M}, \tau \in X$  form a  $\mathcal{L}$ -basis of  $\mathcal{M}$ . It is convenient to modify this basis slightly. For  $\tau \in X$  let  $m_\tau$  be the unique shortest element of the coset  $t_\tau W_f$ . Using the length formula (2.6) one can check that  $m_\tau = t_\tau w_\tau^{-1}$  where  $w_\tau$  is the shortest permutation such that  $w_\tau(\tau) \in -X_+$ . A useful formula for  $w_\tau$  is

$$(3.2) \quad w_\tau(i) = \#\{j = 1, \dots, i \mid \tau_j \leq \tau_i\} + \#\{j = i+1, \dots, n \mid \tau_j < \tau_i\}.$$

The elements

$$(3.3) \quad M_\tau := m_\tau \otimes 1 = v^{-\ell(w_\tau)}(t_\tau \otimes 1)$$

form the *standard basis* of  $\mathcal{M}$ . The action of the generators of  $\mathcal{H}$  in terms of the standard basis is then given by (see [So] §3)

$$(3.4) \quad (H_i + v)(M_\tau) = \begin{cases} M_{s_i(\tau)} + vM_\tau & \text{if } \tau_i > \tau_{i+1} \\ (v + v^{-1})M_\tau & \text{if } \tau_i = \tau_{i+1} , \\ M_{s_i(\tau)} + v^{-1}M_\tau & \text{if } \tau_i < \tau_{i+1} \end{cases} \quad i = 1, \dots, n-1$$

$$(3.5) \quad \omega(M_\tau) = M_{\omega(\tau)}.$$

The Bruhat order on  $W$  induces an order relation on  $X$  by defining

$$(3.6) \quad \tau \leq \eta \iff m_\tau \leq m_\eta.$$

It has the properties

$$(3.7) \quad s_\alpha(\tau) \geq \tau \iff \alpha(\tau) \geq 0 \quad \text{for all } \alpha \in \Delta^+$$

$$(3.8) \quad \tau \leq \eta \iff \omega(\tau) \leq \omega(\eta)$$

$$(3.9) \quad \tau \leq \eta \iff \min\{\tau, s_\alpha(\tau)\} \leq s_\alpha(\eta) \quad \text{for all } \alpha \in \Sigma \text{ with } \alpha(\eta) \leq 0.$$

$$(3.10) \quad \tau \leq 0 \iff \tau = 0.$$

Observe that these properties allow to compute the Bruhat order algorithmically. In fact, with (3.9) one can “move”  $\eta$  into the fundamental alcove. Then, using (3.8), one reduces to  $\eta = 0$ . Then one concludes with (3.10).

In general it is not true that  $\tau \leq \eta$  implies  $w\tau \leq w\eta$  but there is an important special case when this holds:

**3.1. Lemma.** *Let  $\tau, \eta \in X$ ,  $w \in W_f$  and assume that  $\eta - \tau \in \mathbb{Z}\alpha^\vee$  for some  $\alpha \in \Delta_f^+$ . Assume moreover  $w\alpha > 0$ . Then  $\tau \leq \eta$  if and only if  $w\tau \leq w\eta$ .*

*Proof:* Let  $\tau := \eta - k\alpha^\vee$  and  $N := \alpha(\eta)$ . Then  $\tau \leq \eta$  if and only if

$$(3.11) \quad k = \begin{cases} 0, \dots, N-1 & \text{for } N > 0 \\ N, \dots, 0 & \text{for } N \leq 0 \end{cases}$$

See, e.g., [Kn2] Lemma 4.1. The result follows since  $w\alpha(w\eta) = \alpha(\eta)$ . □

#### 4. The Bernstein presentation

For  $\tau \in X_+$  let  $X^\tau := H_{t_\tau}$ . If  $\tau, \eta \in X_+$ , then (2.6) and (2.14) imply

$$(4.1) \quad X^\tau X^\eta = X^{\tau+\eta} = X^\eta X^\tau.$$

Hence we can extend the definition for  $X^\tau$  to all  $\tau \in X$  by

$$(4.2) \quad X^\tau := X^{\tau'} (X^{\tau''})^{-1} \quad \text{where } \tau', \tau'' \in X_+ \text{ with } \tau = \tau' - \tau''.$$

Since  $t_{e_1} = \omega^{-1} s_{n-1} \dots s_1$  is a reduced expression this means concretely in our situation  $X^\tau = X_1^{\tau_1} \dots X_n^{\tau_n}$  with

$$(4.3) \quad X_i := X^{e_i} = H_{i-1}^{-1} \dots H_1^{-1} \omega^{-1} H_{n-1} \dots H_i$$

This way we get a homomorphism

$$(4.4) \quad \Phi : \mathcal{L}[X] := \mathcal{L}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \hookrightarrow \mathcal{H} : x_i \mapsto X_i.$$

The relations between the  $X^\tau$  and the “finite”  $H_i$  (i.e.  $i > 0$ ) have been determined by Bernstein (see [Lu3] 0.3):

$$(4.5) \quad H_i \Phi(\xi) - \Phi(s_i(\xi)) H_i = (v^{-1} - v) \Phi(x_i \frac{\xi - s_i(\xi)}{x_i - x_{i+1}}), \quad i = 1, \dots, n-1.$$

The homomorphism  $\Phi$  also identifies  $\mathcal{L}[X]$  with  $\mathcal{M}$  and we get:

$$(4.6) \quad \Psi : \mathcal{L}[X] \xrightarrow{\sim} \mathcal{M} : \xi \mapsto \Phi(\xi) \otimes 1 = \Phi(\xi)(M_0).$$

By transport of structure, we get an action of  $\mathcal{H}$  on  $\mathcal{L}[X]$ . Concretely, the generators  $H_1, \dots, H_{n-1}, X_1, \dots, X_n$  act as

$$(4.7) \quad H_i(\xi) = v^{-1} s_i(\xi) + (v^{-1} - v) x_i \frac{\xi - s_i(\xi)}{x_i - x_{i+1}}, \quad i = 1, \dots, n-1;$$

$$(4.8) \quad X_i(\xi) = x_i \xi, \quad i = 1, \dots, n.$$

The action of  $\omega$  and  $H_0$  is more complicated and is deduced from the relations

$$(4.9) \quad \omega^{-1} = H_1 \dots H_{n-1} X_n = X_1 H_1^{-1} \dots H_{n-1}^{-1}.$$

$$(4.10) \quad H_0 = \omega^{-1} H_{n-1} \omega = \omega H_1 \omega^{-1} = X_1 X_n^{-1} H_{(1n)}^{-1} \quad \text{with } (1n) := s_1 \dots s_{n-1} \dots s_1.$$

**4.1. Lemma.** *For every  $w \in W_f$  and  $\tau \in X$  holds*

$$(4.11) \quad H_w(x^\tau) \in v^{\ell(w)-2k} x^{w(\tau)} + \sum_{\eta < w(\tau)} \mathcal{L} x^\eta.$$

where

$$(4.12) \quad k := \#\{\alpha \in \Delta_f^+ \mid w\alpha < 0, \alpha(\tau) \geq 0\}.$$

*Proof:* If  $w = 1$ , the statement is trivial. Otherwise write  $w = vs$  with  $s$  a simple reflection and  $\ell(v) < \ell(w)$ . This means  $v\alpha_s > 0$ . Let  $x^\mu$  be a monomial occurring in  $H_s(x^\tau)$  and  $x^\eta$  a monomial occurring in  $H_v(x^\mu)$ . By the explicit formula (4.7) and (3.11) we get  $\mu \leq s(\tau)$  and  $s(\tau) - \mu \in \mathbb{Z}\alpha_s^\vee$ . Then Lemma 3.1 implies  $v(\mu) \leq w(\tau)$ . By induction, we have  $\eta \leq v(\mu)$ , hence  $\eta \leq w(\tau)$ . Finally, the coefficient of  $x^{s\tau}$  in  $H_s(x^\tau)$  is  $v^{-1}$  or  $v$  according to  $\alpha_s(\tau) \geq 0$  or  $\alpha_s(\tau) < 0$ , respectively. On the other hand,  $k = k_w(\tau)$  satisfies the recursion

$$(4.13) \quad k_w(\tau) = k_v(s\tau) + \begin{cases} 1 & \text{if } \alpha_s(\tau) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

which implies the claim on the leading coefficient.  $\square$

## 5. The Kazhdan-Lusztig basis

The defining relations (2.10)–(2.13) of  $\mathcal{H}$  imply that it admits a unique ring automorphism  $d$  with

$$(5.1) \quad d(v) = v^{-1}, \quad d(\omega) = \omega, \quad d(H_i) = H_i^{-1} \quad i = 1, \dots, n.$$

More generally we have  $d(H_w) = H_{w^{-1}}^{-1}$  for any  $w \in W$ . Now put  $\mathcal{H}_{++} := \sum_{x \in W} v\mathbb{Z}[v]H_x$ . Then we have the following fundamental result of Kazhdan-Lusztig ([KL], see also [So]):

**5.1. Theorem.** *For every  $w \in W$  there is a unique  $\underline{H}_w \in \mathcal{H}$  with  $d(\underline{H}_w) = \underline{H}_w$  and  $\underline{H}_w \in H_w + \mathcal{H}_{++}$ . This element is triangular with respect to the Bruhat order, i.e.,  $\underline{H}_w \in \sum_{v \leq w} \mathcal{L}H_v$ . Moreover, the collection of  $\underline{H}_w$ ,  $w \in W$  forms an  $\mathcal{L}$ -basis of  $\mathcal{H}$ .*

A similar construction works for  $\mathcal{M}$ . Since the homomorphism  $\mathcal{H}_f \rightarrow \mathcal{L}$  defining  $\mathcal{L}(v^{-1})$  commutes with  $d$  we may define an involution of  $\mathcal{M}$ , also denoted by  $d$ , by

$$(5.2) \quad d(\xi \otimes a(v)) := d(\xi) \otimes a(v^{-1}).$$

Again we form  $\mathcal{M}_{++} := \sum_{\tau \in X} v\mathbb{Z}[v]M_\tau$  and obtain (see [So]):

**5.2. Theorem.** *For every  $\tau \in X$  there is a unique  $\underline{M}_\tau \in \mathcal{M}$  with  $d(\underline{M}_\tau) = \underline{M}_\tau$  and  $\underline{M}_\tau \in M_\tau + \mathcal{M}_{++}$ . This element is triangular with respect to the Bruhat order, i.e.,  $\underline{M}_\tau \in \sum_{\eta \leq \tau} \mathcal{L}M_\eta$ . Moreover, the collection of  $\underline{M}_\tau$ ,  $\tau \in X$  forms an  $\mathcal{L}$ -basis of  $\mathcal{M}$ .*

**Remark:** Observe that the triangularity property of  $\underline{M}_\tau$  implies easily that also  $d$  is triangular, i.e.,  $d(M_\tau) \in \sum_{\eta \leq \tau} \mathcal{L}M_\eta$ .

Using the bijection  $\Psi : \mathcal{L}[X] \xrightarrow{\sim} \mathcal{M}$  the involution  $d$  may be transported to  $\mathcal{L}[X]$ , which we also denote by  $d$ . Explicitly we get ([Kn2] Lemma 3.4):

**5.3. Theorem.** *Let  $p \mapsto \bar{p}$  be the ring involution of  $\mathcal{L}[X]$  with  $\bar{v} = v^{-1}$  and  $\bar{x}_i = x_i$ . Let  $w_0 \in W_f$  be the longest element. Then for any  $p \in \mathcal{L}[X]$  holds*

$$(5.3) \quad d(\Phi(p)) = H_{w_0} \Phi(w_0 \bar{p}) H_{w_0}^{-1}.$$

*In particular,*

$$(5.4) \quad d(p) = v^{\ell(w_0)} H_{w_0}(w_0 \bar{p}).$$

We are going to need only two properties of the Kazhdan-Lusztig basis.

**5.4. Lemma.** *Let  $\tau \in X$  and  $s \in \Sigma$  with  $s(\tau) \leq \tau$ . Then  $H_s(\underline{M}_\tau) = v^{-1} \underline{M}_\tau$ .*

For a proof see, e.g., [So] Prop. 3.6. The Lemma implies in particular that  $\Psi^{-1}(\underline{M}_\tau)$  is symmetric whenever  $\tau \in -X_+$ . In fact, it can be computed explicitly:

**5.5. Theorem.** *For  $\lambda \in X_+$  let  $s_\lambda$  be the corresponding Schur polynomial. Then*

$$(5.5) \quad \underline{M}_{-\lambda} = \Psi(s_\lambda(x_1^{-1}, \dots, x_n^{-1}))$$

This result is due to Lusztig, first proved for  $A_{n-1}$  in [Lu1] and then for arbitrary root systems in [Lu2]. For an alternate proof see [Kn2]. It is the key to our approach to Kostka polynomials.

## 6. Macdonald polynomials

As mentioned, the action (4.9) of  $\omega$  on  $\mathcal{L}[X]$  is quite complicated. Cherednik had the idea (see e.g. [Ch]) to replace  $\omega$  by

$$(6.1) \quad \tilde{\omega}(f)(x_1, \dots, x_n) := f(qx_n, x_1, \dots, x_{n-1})$$

where  $q$  is an additional parameter. This formula is motivated by the affine linear action (2.8) of  $\omega$ . Also the action of  $H_0$  becomes easy this way:

$$(6.2) \quad \tilde{H}_0 := \tilde{\omega} H_1 \tilde{\omega}^{-1} = v^{-1} \tilde{s}_0 + (v^{-1} - v) x_n \frac{1 - \tilde{s}_0}{x_n - q^{-1} x_1}$$

with

$$(6.3) \quad \tilde{s}_0(p) := p(qx_n, x_2, \dots, x_{n-1}, q^{-1}x_1).$$



One checks that  $\tilde{H}_0, H_1, \dots, H_{n-1}, \tilde{\omega}$  satisfy the relations (2.10)–(2.13) and therefore generate another copy  $\tilde{\mathcal{H}}$  of  $\mathcal{H}$ . In particular,  $\tilde{\mathcal{H}}$  will contain a copy of  $\mathcal{L}[X]$  which we choose to be generated by the elements

$$(6.4) \quad \xi_i := v^{1-n} H_{i-1} \dots H_1 \tilde{\omega}^{-1} H_{n-1}^{-1} \dots H_i^{-1}, \quad i = 1, \dots, n$$

Note that this definition is “dual” to (4.3) and also has the factor  $v^{1-n}$ . The reason for this is to get later the stability property (9.20). The main feature of  $\tilde{\mathcal{H}}$  is that it acts locally finitely on  $\mathcal{L}_q[X]$  where  $\mathcal{L}_q := \mathcal{L}[q, q^{-1}]$ . More precisely,

**6.1. Lemma.** *For  $i = 1, \dots, n$  and  $\tau \in X$  holds*

$$(6.5) \quad \xi_i(x^\tau) \in q^{-\tau_i} t^{1-w_\tau(i)} x^\tau + \sum_{\mu < \tau} \mathcal{L}_q x^\mu.$$

*Proof:* First, we show that the  $\xi_i$  are triangular with respect to the Bruhat order. It suffices to do this for  $\Xi_i := \xi_i \xi_{i+1} \dots \xi_n$  with  $i = 1, \dots, n$ . Formula (6.4) implies  $\Xi_i = v^{-N} H_{w_i} \tilde{\omega}^{i-n-1}$  where  $w_i(\tau) = (\tau_{i+1}, \dots, \tau_n, \tau_1, \dots, \tau_i)$  and  $N = (n-i+1)(n-1)$ . Thus,

$$(6.6) \quad \Xi_i(x^\tau) = v^{-N} H_{w_i} \tilde{\omega}^{i-n-1}(x^\tau) = q^{-\tau_i - \dots - \tau_n} v^{-N} H_{w_i} x^{w_i^{-1}(\tau)} \in \sum_{\mu \leq \tau} \mathcal{L}_q x^\mu$$

by Lemma 4.1. The formula for the leading coefficient follows easily from (4.11), (4.12), and (3.2).  $\square$

Since the Cherednik operators  $\xi_1, \dots, \xi_n$  commute and are triangular with distinct diagonal terms they have a common eigenbasis, the *non-symmetric Macdonald polynomials*  $\mathcal{E}_\lambda$ .

**6.2. Corollary.** *For every  $\lambda \in X$  there is  $\mathcal{E}_\lambda \in \mathcal{L}_q[X]$ , unique up to a scalar, with*

$$(6.7) \quad \xi_i(\mathcal{E}_\lambda) = q^{\lambda_i} t^{1-w_{-\lambda}(i)} \mathcal{E}_\lambda, \quad i = 1, \dots, n$$

*Moreover,  $\mathcal{E}_\lambda$  is triangular with respect to the Bruhat order:*

$$(6.8) \quad \mathcal{E}_\lambda \in \sum_{\mu \leq -\lambda} \mathcal{L}_q x^\mu.$$

**Remark:** Usually (see, e.g., [M3] (2.7.5)), the triangularity of  $\mathcal{E}_\lambda$  is expressed with respect to an order which is finer than the Bruhat order.

We are normalizing  $\mathcal{E}_\lambda$  in the following way. As usual, we represent  $\lambda$  by its *diagram*, i.e., the set of pairs  $(i, j) \in \mathbb{Z}^2$  (called *boxes*) with  $1 \leq j \leq \lambda_i$ . To a box  $s = (i, j) \in \lambda$  we associate its *arm-length*

$$(6.9) \quad a_\lambda(s) := \lambda_i - j$$

and its *leg-length*

$$(6.10) \quad l_\lambda(s) := \#\{k < i \mid j \leq \lambda_k + 1 \leq \lambda_i\} + \#\{k > i \mid j \leq \lambda_k \leq \lambda_i\}.$$

Now we demand that the coefficient of  $x^{-\lambda}$  in  $\mathcal{E}_\lambda$  is

$$(6.11) \quad \prod_{s \in \lambda} \left(1 - q^{a_\lambda(s)+1} t^{l_\lambda(s)+1}\right).$$

One can show, [Kn1] Cor. 5.2, that with this normalization the coefficients of  $\mathcal{E}_\lambda$  are polynomials in  $q$  and  $t$ .

## 7. The polynomial part of $\mathcal{M}$

Subsequently, we are only interested in the “polynomial” part of  $\mathcal{M}$ . The reason for this is its stability properties as  $n \rightarrow \infty$ . Let us first introduce the polynomial part  $\mathcal{H}^{\text{pol}}$  of  $\mathcal{H}$ , namely the subalgebra generated by  $\mathcal{H}_f$ , and  $\omega$  (but *not*  $\omega^{-1}$ ). Let  $\Lambda := \mathbb{N}^n \subseteq X$  and consider the submonoid  $W^{\text{pol}} := W_f \ltimes (-\Lambda)$  of  $W$ . Then we have:

**7.1. Theorem.** *The set  $\{H_w \mid w \in W^{\text{pol}}\}$  is an  $\mathcal{L}$ -basis of  $\mathcal{H}^{\text{pol}}$ . Moreover,  $Z_i := X_i^{-1} \in \mathcal{H}^{\text{pol}}$  for all  $i$  and*

$$(7.1) \quad \mathcal{L}[Z_1, \dots, Z_n] \otimes_{\mathcal{L}} \mathcal{H}_f \rightarrow \mathcal{H}^{\text{pol}} : p(Z) \otimes u \mapsto p(Z)u$$

*is bijective. Furthermore,  $\mathcal{H}^{\text{pol}}$  has the following presentations:*

*i) It is generated by the subalgebras  $\mathcal{H}_f$  and  $\mathcal{L}[Z_1, \dots, Z_n]$  with relations*

$$(7.2) \quad H_i p - (s_i p) H_i = (v - v^{-1}) Z_{i+1} \frac{p - (s_i p)}{Z_i - Z_{i+1}} \text{ for } i = 1, \dots, n-1, p \in \mathcal{L}[Z_1, \dots, Z_n].$$

*ii) It is generated by the subalgebra  $\mathcal{H}_f$  and  $Z_1, \dots, Z_n$  with relations*

$$(7.3) \quad H_i Z_i H_i = Z_{i+1} \text{ for } i = 1, \dots, n-1$$

$$(7.4) \quad H_j Z_i = Z_i H_j \text{ for } i = 1, \dots, n, j = 1, \dots, n-1, i-j \neq 0, 1$$

$$(7.5) \quad Z_i Z_j = Z_j Z_i \text{ for } i, j = 1, \dots, n$$

*iii) It is generated by the subalgebra  $\mathcal{H}_f$  and  $Z_1$  with relations*

$$(7.6) \quad H_1 Z_1 H_1 Z_1 = Z_1 H_1 Z_1 H_1$$

$$(7.7) \quad H_i Z_1 = Z_1 H_i \text{ for } i = 2, \dots, n$$

iv) It is generated by the subalgebra  $\mathcal{H}_f$  and  $\omega$  with relations

$$(7.8) \quad H_i \omega = \omega H_{i+1} \text{ for } i = 1, \dots, n-2$$

$$(7.9) \quad H_{n-1} \omega^2 = \omega^2 H_1$$

*Proof:* Formula (4.3) implies  $Z_i \in \mathcal{H}^{\text{pol}}$ . Let  $w_0 \in W_f$  be the longest element. Then, for  $\lambda \in \Lambda \cap X_+$

$$(7.10) \quad \mathcal{H}^{\text{pol}} \ni Z^\lambda = (X^\lambda)^{-1} = H_{t_\lambda}^{-1} = d(H_{t_{-\lambda}}) = H_{w_0} H_{t_{-w_0(\lambda)}} H_{w_0}^{-1}$$

(by (5.3)). This implies  $H_{t_{-\lambda}} \in \mathcal{H}^{\text{pol}}$  for all  $\lambda \in \Lambda \cap X_+$ . Since  $H_{sw} = H_s^{\pm 1} H_w$  and  $H_{ws} = H_w H_s^{\pm 1}$  for all simple reflections  $s$  we get  $\mathcal{H}' \subseteq \mathcal{H}^{\text{pol}}$  where  $\mathcal{H}'$  is the  $\mathcal{L}$ -submodule spanned by all  $H_w \in \mathcal{H}^{\text{pol}}$ ,  $w \in W^{\text{pol}}$ . Conversely, formula (2.14) implies  $\omega \mathcal{H}' \subseteq \mathcal{H}'$ . Hence  $\mathcal{H}^{\text{pol}} \subseteq \mathcal{H}'$ .

i) Let now  $\mathcal{H}'$  be the algebra generated by  $\mathcal{L}[Z]$  and  $\mathcal{H}_f$  subject to relation (7.2). Then there are natural maps

$$(7.11) \quad \begin{array}{ccccc} \mathcal{L}[Z] \otimes_{\mathcal{L}} \mathcal{H}_f & \xrightarrow{\varphi_1} & \mathcal{H}' & \xrightarrow{\varphi_2} & \mathcal{H}^{\text{pol}} \\ \downarrow \varphi_3 & & & & \downarrow \varphi_4 \\ \mathcal{L}[Z, Z^{-1}] \otimes_{\mathcal{L}} \mathcal{H}_f & \xrightarrow{\varphi_5} & & & \mathcal{H} \end{array}$$

Since  $\varphi_3$  and  $\varphi_5$  are injective, also  $\varphi_1$  is injective. Relation (7.2) implies that  $\varphi_1$  is also surjective. Thus,  $\varphi_2$  is injective. Now  $\omega = H_{n-1} \dots H_1 Z_1$  implies that  $\varphi_2$  is also surjective. This implies the bijectivity of (7.1).

ii) Relation (7.5) simply means that  $\mathcal{H}^{\text{pol}}$  contains  $\mathcal{L}[Z]$  as a subalgebra. Moreover (7.3), (7.4) are equivalent to (7.2) for  $p = Z_j$ . It is well known (see [Lu3] 3.6) that that case implies (7.2) for any  $p$ . Thus, the presentation i) and ii) are equivalent.

iii) Here, we are defining  $Z_2, \dots, Z_n$  using formula (7.3). Then (7.6) is nothing else than  $Z_2 Z_1 = Z_1 Z_2$ . Thus, ii) implies iii). Conversely, assume (7.6), (7.7) hold. Relations (7.3) are true by definition. Relation (7.4) follows easily for  $i < j$ . For  $i = j + 2$  we have

$$(7.12) \quad \begin{aligned} H_j Z_i &= H_j H_{j+1} H_j Z_j H_j H_{j+1} = H_{j+1} H_j H_{j+1} Z_j H_j H_{j+1} = \\ &= H_{j+1} H_j Z_j H_{j+1} H_j H_{j+1} = H_{j+1} H_j Z_j H_j H_{j+1} H_j = Z_i H_j. \end{aligned}$$

For  $i > j + 2$  we get (7.4) by induction from  $Z_i = H_{i-1} Z_{i-1} H_{i-1}$ . Finally, (7.5) follows the same way by induction from (7.4) and (7.6).

iv) First assume the relations in iv). Define  $Z_1 := H_1^{-1} \dots H_{n-1}^{-1} \omega$ . Then

$$(7.13) \quad H_1 Z_1 H_1 Z_1 = H_2^{-1} \dots H_{n-1}^{-1} \omega H_2^{-1} \dots H_{n-1}^{-1} \omega =$$

$$(7.14) \quad = (H_2^{-1} \dots H_{n-1}^{-1} H_1^{-1} \dots H_{n-2}^{-1} H_{n-1}^{-1}) H_{n-1} \omega^2$$

$$(7.15) \quad Z_1 H_1 Z_1 H_1 = H_1^{-1} \dots H_{n-1}^{-1} \omega H_2^{-1} \dots H_{n-1}^{-1} \omega H_1 =$$

$$(7.16) \quad = (H_1^{-1} \dots H_{n-1}^{-1} H_1^{-1} \dots H_{n-2}^{-1}) \omega^2 H_1$$

Both expressions are equal since the terms in parenthesis correspond to reduced expressions of the same permutation namely  $(n, n-1, 1, \dots, n-2)$ . Moreover, for  $i \geq 2$  we have

$$(7.17) \quad \begin{aligned} H_i^{-1} Z_1 &= H_1^{-1} \dots H_{i-2}^{-1} (H_i^{-1} H_{i-1}^{-1} H_i^{-1}) H_{i+1}^{-1} \dots H_{n-1}^{-1} \omega = \\ &= H_{i-2}^{-1} (H_{i-1}^{-1} H_i^{-1} H_{i-1}^{-1}) H_{i+1}^{-1} \dots H_{n-1}^{-1} \omega = Z_1 H_i^{-1}. \end{aligned}$$

Assume now conversely that relations *ii)* hold. We define  $\omega := H_{n-1} \dots H_1 Z_1$ . Then

$$(7.18) \quad \omega H_{i+1} = H_{n-1} \dots H_{i+1} H_i H_{i+1} H_{i-1} \dots H_1 Z_1 = H_i \omega$$

Finally, (7.9) can be deduced from the equality of (7.14) and (7.16). □

**Remark:** The proof shows that *ii)–iv)* are also equivalent presentations of a “braid monoid” with generators  $H_1^{\pm 1}, \dots, H_n^{\pm 1}$  and  $\omega$ . Of course, the braid relations among the  $H_i$  should also hold.

Now let  $\mathcal{M}^{\text{pol}}$  be the  $\mathcal{L}$ -submodule of  $\mathcal{M}$  spanned by all  $M_\lambda$  with  $\lambda \in -\Lambda$ .

**7.2. Theorem.** *The map*

$$(7.19) \quad \mathcal{H}^{\text{pol}} \otimes_{\mathcal{H}_f} \mathcal{L}(v^{-1}) \rightarrow \mathcal{M}$$

*is injective with image  $\mathcal{M}^{\text{pol}}$ . In particular,  $\mathcal{M}^{\text{pol}}$  is an  $\mathcal{H}^{\text{pol}}$ -module and*

$$(7.20) \quad \Psi : \mathcal{L}[z_1, \dots, z_n] \xrightarrow{\sim} \mathcal{M}^{\text{pol}}$$

*is an isomorphism where  $z_i := x_i^{-1}$ .*

*Proof:* Consider the following commutative diagram:

$$(7.21) \quad \begin{array}{ccc} \mathcal{L}[z] & \xrightarrow{\varphi_1} & \mathcal{H}^{\text{pol}} \otimes_{\mathcal{H}_f} \mathcal{L}(v^{-1}) \\ \downarrow \varphi_2 & & \downarrow \varphi_3 \\ \mathcal{L}[z, z^{-1}] & \xrightarrow{\varphi_4} & \mathcal{M} \end{array}$$

Then  $\varphi_1, \varphi_4$  are bijective by (7.1), (4.6), respectively, while  $\varphi_2$  is obviously injective. Hence,  $\varphi_3$  is injective. Formulas (3.4), (3.5) show that  $\mathcal{M}^{\text{pol}}$  is an  $\mathcal{H}^{\text{pol}}$ -module. Hence  $\text{Im } \varphi_3 \subseteq \mathcal{M}^{\text{pol}}$ . The converse inclusion follows from (3.3). □

Observe that the operators  $H_i, i = 1, \dots, n-1$  and  $\tilde{\omega}$  take the following form in the coordinates  $z_i = x_i^{-1}$ :

$$(7.22) \quad H_i = v^{-1} s_i + (v - v^{-1}) z_{i+1} \frac{1 - s_i}{z_i - z_{i+1}}$$

$$(7.23) \quad \tilde{\omega}(f)(z_1, \dots, z_n) = f(q^{-1}z_n, z_1, \dots, z_{n-1})$$

Moreover, to simplify notation, we write from now on for  $\lambda \in \Lambda$

$$(7.24) \quad M^\lambda := M_{-\lambda}, \quad \underline{M}^\lambda := \underline{M}_{-\lambda}, \quad w^\lambda := w_{-\lambda}.$$

Observe that  $w^\lambda$  is the shortest permutation such that  $w^\lambda(\lambda)$  is a partition. We also modify  $\omega$ :

$$(7.25) \quad \omega^*(\tau) := -\omega(-\tau) = (\tau_2, \dots, \tau_n, \tau_1 + 1) \quad \text{hence} \quad \omega(M^\lambda) = M^{\omega^*(\lambda)}$$

Finally, the modified Bruhat order is

$$(7.26) \quad \lambda \preceq \mu \iff -\lambda \leq -\mu.$$

**7.3. Lemma.** *Let  $\lambda \in X$  and  $\mu \in \Lambda$ . Then  $\lambda \preceq \mu$  implies  $\lambda \in \Lambda$ .*

*Proof:* Let  $\lambda, \mu$  be a counterexample with  $N := |\mu| = \sum_i \mu_i$  minimal. Let  $w \in W_f$  be minimal with  $\mu' := w\mu \in -X_+$ , i.e.,  $\mu'_1 \leq \dots \leq \mu'_n$ . By (3.7), (3.9) there is  $w' \in W_f$ ,  $w' \leq w$  with  $\lambda' := w'\lambda \preceq \mu'$ . Clearly also  $\lambda' \notin \Lambda$  and  $\mu \neq 0$ . Hence  $\mu'_n > 0$  which implies that  $\mu'' := (\omega^*)^{-1}(\mu') = (\mu'_n - 1, \mu'_1, \dots, \mu'_{n-1}) \in \Lambda$  while  $\lambda'' := (\omega^*)^{-1}(\lambda') \notin \Lambda$ . Since  $\lambda'' \preceq \mu''$  and  $\sum_i \mu''_i = N - 1$  we get a contradiction to the minimality of  $N$ .  $\square$

**7.4. Corollary.** *The subset  $\mathcal{M}^{\text{pol}}$  is stable under the involution  $d$ . Moreover, the Kazhdan-Lusztig elements  $\underline{M}^\lambda$  with  $\lambda \in \Lambda$  form an  $\mathcal{L}$ -basis of  $\mathcal{M}^{\text{pol}}$ .*

*Proof:* This follows from Lemma 7.3 and the triangularity of the involution  $d$  and the Kazhdan-Lusztig elements.  $\square$

**7.5. Corollary.** *For  $\lambda \in \Lambda$ , the Macdonald polynomial  $\mathcal{E}_\lambda$  is in  $\mathcal{M}_q^{\text{pol}} := \mathcal{M}^{\text{pol}} \otimes_{\mathcal{L}} \mathcal{L}_q$ .*

*Proof:* Follows immediately from Lemma 7.3 and the triangularity property (6.8).  $\square$

## 8. Recursion formulas for Macdonald polynomials

In this section, we describe the recursion formulas from [Kn1]<sup>1</sup> which produce exactly the  $\mathcal{E}_\lambda$  with  $\lambda \in \Lambda$ . For  $m = 1, \dots, n$  we define the operators

$$(8.1) \quad \tilde{\Phi}_m := H_m \dots H_{n-1} Z_n \tilde{\omega},$$

<sup>1</sup> For the convenience of the reader we include following conversion table between notations:

[Kn1]	$t$	$H_i$	$\bar{H}_i$	$\Delta$	$\xi_i$	$\mathcal{E}_\lambda$	$\Phi$	$A_m$	$\bar{A}_m$	$\Phi'$	$A'_m$	$\bar{A}'_m$
This paper	$t = v^2$	$vH_i^{-1}$	$vH_i$	$\tilde{\omega}$	$\xi_i$	$\mathcal{E}_\lambda$	$Z_n \tilde{\omega}$	$v^{n-m} \bar{\Phi}_m$	$v^{n-m} \Phi_m$	$v^{1-n} \omega$	$v^{1-m} \bar{\Phi}_m$	$v^{1-m} \Phi_m$

$$(8.2) \quad \bar{\Phi}_m := H_m^{-1} \dots H_{n-1}^{-1} Z_n \tilde{\omega}.$$

Recall that the length  $l(\lambda)$  of  $\lambda \in \Lambda = \mathbb{N}^n$  is the maximal  $m \geq 0$  with  $\lambda_m \neq 0$  (so  $\lambda = 0$  if and only if  $l(\lambda) = 0$ ).

**8.1. Theorem.** ([Kn1] Thm. 5.1) *For  $\lambda \in \Lambda$  let  $m := l(\lambda)$ . Then*

$$(8.3) \quad \mathcal{E}_\lambda = q^{\lambda_m - 1} v^{n-m} \left( \tilde{\Phi}_m - q^{\lambda_m} v^{2a} \bar{\Phi}_m \right) (\mathcal{E}_{\lambda^*}) \quad \text{where}$$

$$(8.4) \quad \lambda^* := (\lambda_m - 1, \lambda_1, \dots, \lambda_{m-1}, 0, \dots, 0)$$

$$(8.5) \quad a := 1 + \#\{i = 1, \dots, m \mid \lambda_i < \lambda_m\}.$$

Clearly, starting from  $\mathcal{E}_0 = 1$ , this formula allows to compute  $\mathcal{E}_\lambda$  for all  $\lambda \in \Lambda$  in a unique way.

Following [Kn1], we are going to rewrite the recursion (8.3). Equations (4.3) and (6.4) imply

$$(8.6) \quad Z_n \tilde{\omega} = v^{1-n} \omega \xi_1^{-1}.$$

Inserting this into (8.1), (8.2) and observing that  $\mathcal{E}_{\lambda^*}$  is an eigenvector for  $\xi_1^{-1}$  (see (6.7)) we obtain

$$(8.7) \quad \mathcal{E}_\lambda = v^{m+1-2a} \left( \Phi_m - q^{\lambda_m} v^{2a} \bar{\Phi}_m \right) (\mathcal{E}_{\lambda^*})$$

with the new operators

$$(8.8) \quad \Phi_m := H_m \dots H_{n-1} \omega = H_{c_m},$$

$$(8.9) \quad \bar{\Phi}_m := H_m^{-1} \dots H_{n-1}^{-1} \omega = H_{c_m^{-1}}^{-1} = d(H_{c_m})$$

where

$$(8.10) \quad c_m := s_{m-1} \dots s_1 t_{-e_1} : \lambda \mapsto (\lambda_2, \dots, \lambda_m, \lambda_1 - 1, \lambda_{m+1}, \dots, \lambda_n)$$

For the renormalized Macdonald polynomial

$$(8.11) \quad \tilde{\mathcal{E}}_\lambda = v^{\ell(w^\lambda)} \mathcal{E}_\lambda$$

we obtain the simple formula

$$(8.12) \quad \tilde{\mathcal{E}}_\lambda = \left( \Phi_m - q^{\lambda_m} t^a \bar{\Phi}_m \right) (\tilde{\mathcal{E}}_{\lambda^*}).$$

The big advantage of (8.7) and (8.12) over (8.3) is that the parameter  $q$  is not involved in the operators  $\Phi_m$  and  $\bar{\Phi}_m$ .

The first application of the recursion formulas is the following integrality result from [Kn1]:

**8.2. Theorem.** *For every  $\lambda \in \Lambda$  holds  $\mathcal{E}_\lambda \in \mathbb{Z}[t, q][z_1, \dots, z_n]$ .*

*Proof:* The definitions (8.1), (8.2) and formula (7.22) show that the operators  $v^{n-m}\tilde{\Phi}_m$  and  $v^{n-m}\bar{\Phi}_m$  preserve the ring  $\mathbb{Z}[t, q, q^{-1}][z_1, \dots, z_n]$ . Thus, formula (8.3) implies that  $\mathcal{E}_\lambda$  is in this ring. On the other hand, (8.7) implies clearly the non-occurrence of negative powers of  $q$ .  $\square$

When we express  $\mathcal{E}_\lambda$  in terms of the standard basis then we get

**8.3. Corollary.**  $\Psi(\mathcal{E}_\lambda) \in \sum_{\mu \in \Lambda} \mathbb{Z}[v^2, q]v^{-\ell(w^\mu)}M^\mu.$

*Proof:* [Kn2] Lemma 4.2 implies that the transition matrix between monomials  $z^\tau$  and the elements  $v^{-\ell(w^\mu)}\Psi^{-1}(M^\mu)$  is unitriangular with coefficients in  $\mathbb{Z}[t]$ .  $\square$

According to this Corollary, the coefficients of  $\Psi(\mathcal{E}_\lambda)$  might contain arbitrary large negative powers of  $v$ . Computational evidence leads to:

**8.4. Conjecture.**  $\Psi(\tilde{\mathcal{E}}_\lambda) \in \sum_{\mu \in \Lambda} \mathbb{Z}[v, q]M^\mu.$

Another consequence of the recursion formula is that  $\mathcal{E}_\lambda$  is almost selfdual. More precisely, using the isomorphism  $\Psi : \mathcal{L}[Z_i] \xrightarrow{\sim} \mathcal{M}^{\text{pol}}$  we can transport the involution  $d$  to  $\mathcal{L}[Z_i]$  (Theorem 5.3). Then we extend it to  $\mathcal{L}_q[Z_i]$  by defining  $d(q) := q^{-1}$ .

**8.5. Theorem.** *For every  $\lambda \in \Lambda$  holds*

$$(8.13) \quad d(\mathcal{E}_\lambda) = (-1)^{|\lambda|}q^{-A}t^{-B}\mathcal{E}_\lambda, \quad d(\tilde{\mathcal{E}}_\lambda) = (-1)^{|\lambda|}q^{-A}t^{-B-\ell(w^\lambda)}\tilde{\mathcal{E}}_\lambda$$

where

$$(8.14) \quad A := \sum_{i \geq 1} \binom{\lambda_i + 1}{2}$$

$$(8.15) \quad B := \sum_{i \geq 1} i\lambda_i^+ \quad \text{with} \quad \lambda^+ := w^\lambda(\lambda) \in X_+.$$

*Proof:* The formula for  $\tilde{\mathcal{E}}_\lambda$  follows immediately from that for  $\mathcal{E}_\lambda$  and the definition (8.11).

Write  $A(\lambda)$  and  $B(\lambda)$  for  $A$  and  $B$ , respectively. We proceed by induction on  $|\lambda|$ . The assertion is obvious for  $\lambda = 0$ . Now assume it holds for  $\lambda^*$ . Clearly, we have  $d\Phi_m = \bar{\Phi}_m d$ . Hence

$$(8.16) \quad \begin{aligned} d(\mathcal{E}_\lambda) &= v^{-m-1+2a}(\bar{\Phi}_m - q^{-\lambda_m}v^{-2a}\Phi_m)((-1)^{|\lambda^*|}q^{-A(\lambda^*)}v^{-2B(\lambda^*)}\mathcal{E}_{\lambda^*}) = \\ &= (-1)^{|\lambda^*|+1}q^{-A(\lambda^*)-\lambda_m}v^{-2B(\lambda^*)-m-1}(-q^{\lambda_m}v^{2a}\bar{\Phi}_m + \Phi_m)(\mathcal{E}_{\lambda^*}) = \\ &= (-1)^{|\lambda^*|+1}q^{-A(\lambda^*)-\lambda_m}v^{-2B(\lambda^*)-2(m+1-a)}\mathcal{E}_\lambda \end{aligned}$$

Thus we have to show

$$(8.17) \quad |\lambda| = |\lambda^*| + 1,$$

$$(8.18) \quad A(\lambda) = A(\lambda^*) + \lambda_m,$$

$$(8.19) \quad B(\lambda) = B(\lambda^*) + (m + 1 - a).$$

Equation (8.17) is obvious, (8.18) is easy, and it remains to prove (8.19). Let  $\lambda_k^+$  be the rightmost entry of  $\lambda^+$  which equals  $\lambda_m$ . Then

$$(8.20) \quad k = \#\{i = 1, \dots, m \mid \lambda_i \geq \lambda_m\} = m - (a - 1) = m + 1 - a.$$

On the other hand  $(\lambda^*)^+$  differs from  $\lambda^+$  only in its  $k$ -th entry which is  $\lambda_m - 1$ . Hence  $B(\lambda) = B(\lambda^*) + k$  which proves (8.19).  $\square$

## 9. Stabilization

Now we want to study Macdonald and Kazhdan-Lusztig polynomials as  $n \rightarrow \infty$ . The Hecke algebra studied so far will be denoted by  $\mathcal{H}_n$ . Its parabolic module is  $\mathcal{M}_n$  with its polynomial subset  $\mathcal{M}_n^{\text{pol}}$ . The element  $\omega$  of  $\mathcal{H}_n$  will be denoted  $\omega_n$ .

**9.1. Theorem.** *Let  $\pi_n : \mathcal{M}_n^{\text{pol}} \rightarrow \mathcal{M}_{n-1}^{\text{pol}}$  be the projection with*

$$(9.1) \quad \pi(M^\lambda) = \begin{cases} M^{\lambda'} & \text{if } \lambda_n = 0 \text{ and where } \lambda' := (\lambda_1, \dots, \lambda_{n-1}) \\ 0 & \text{if } \lambda_n > 0. \end{cases}$$

*Then the following commutation relations hold:*

$$(9.2) \quad \pi_n H_i = H_i \pi_n \quad i = 1, \dots, n-2$$

$$(9.3) \quad \pi_n \omega_n = 0$$

$$(9.4) \quad \pi_n H_{n-1} \omega_n = \pi_n H_{n-1}^{-1} \omega_n = \omega_{n-1} \pi$$

$$(9.5) \quad \pi_n Z_i = \begin{cases} Z_i \pi_n & \text{for } i = 1, \dots, n-1 \\ 0 & \text{for } i = n \end{cases}$$

*Proof:* Equation (9.2) follows immediately from (3.4). For  $\lambda \in \Lambda$  let  $\lambda^* := \omega_n^*(\lambda) = (\lambda_2, \dots, \lambda_n, \lambda_1 + 1)$ . Then (9.3) follows from  $\omega_n(M^\lambda) = M^{\lambda^*}$ . Moreover

$$(9.6) \quad \pi_n H_{n-1} \omega_n(M^\lambda) = \pi_n H_{n-1} M^{\lambda^*} \quad \begin{cases} = \pi_n M^{s_{n-1}(\lambda^*)} = \omega_{n-1} \pi_n(M^\lambda) & \text{if } \lambda_n = 0 \\ \in \mathcal{L} \pi_n M^{s_{n-1}(\lambda^*)} + \mathcal{L} \pi_n M^{\lambda^*} = 0 & \text{if } \lambda_n > 0 \end{cases}$$

This proves the first part of (9.4). The second part follows using (9.3). Finally, we get (9.5) by using the above and the explicit expression (4.3) for  $Z_i = X_i^{-1}$ .  $\square$



Let  $\pi_n : \mathcal{L}[z_1, \dots, z_{n-1}, z_n] \rightarrow \mathcal{L}[z_1, \dots, z_{n-1}]$  be the obvious projection. Then equation (9.5) implies:

**9.2. Corollary.** *The following diagram commutes*

$$(9.7) \quad \begin{array}{ccc} \mathcal{L}[z_1, \dots, z_{n-1}, z_n] & \xrightarrow{\Psi_n} & \mathcal{M}_n^{\text{pol}} \\ \downarrow \pi_n & & \downarrow \pi_n \\ \mathcal{L}[z_1, \dots, z_{n-1}] & \xrightarrow{\Psi_{n-1}} & \mathcal{M}_{n-1}^{\text{pol}} \end{array}$$

Both  $\mathcal{L}[z_1, \dots, z_{n-1}, z_n]$  and  $\mathcal{M}_n^{\text{pol}}$  carry a natural grading, the first by degree, the second by defining  $\deg M^\lambda := |\lambda| = \sum_i \lambda_i$ . Moreover,  $\Psi_n$  is degree-preserving. This follows from the definition (4.3) of  $Z_i = X_i^{-1}$  and the fact that  $H_i, \omega$  is homogeneous of degree 0 and 1, respectively (see (3.4), (3.5)).

Corollary 9.2 implies that if we consider the projective limits

$$(9.8) \quad \mathcal{M}_\infty^{\text{pol}} := \varprojlim \mathcal{M}_n^{\text{pol}}, \quad \mathcal{P}_\infty := \varprojlim \mathcal{L}[z_1, \dots, z_n]$$

in the category of graded abelian groups, then we get an isomorphism

$$(9.9) \quad \Psi : \mathcal{P}_\infty \xrightarrow{\sim} \mathcal{M}_\infty^{\text{pol}}$$

More precisely, let  $\Lambda := \mathbb{N}^{(\infty)}$  be the set of all sequences of natural numbers almost all of which are zero. Then  $\mathcal{M}_\infty^{\text{pol}}, \mathcal{P}_\infty$  is the set of all possibly infinite sums  $\sum_\lambda a_\lambda M^\lambda, \sum_\lambda a_\lambda z^\lambda$ , respectively where  $\lambda$  runs through a subset of  $\Lambda$  in which  $|\lambda|$  remains bounded.

A further consequence of Theorem 9.1 is

**9.3. Corollary.** *The space  $\mathcal{M}_\infty^{\text{pol}}$  carries an action of the operators  $H_i, Z_i, \Phi_i, \bar{\Phi}_i$  ( $i \geq 1$ ).*

Next we need a property of the Bruhat order:

**9.4. Lemma.** *Fix an  $i$  with  $1 \leq i \leq n+1$ . For  $\lambda \in \mathbb{Z}^{n+1}$  let  $\lambda' \in \mathbb{Z}^n$  be obtained from  $\lambda$  by omitting the  $i$ -th entry. Let  $\lambda, \mu \in \mathbb{Z}^n$  with  $\lambda_i = \mu_i$ . Then  $\lambda \leq \mu$  if and only if  $\lambda' \leq \mu'$ .*

*Proof:* First, by applying  $\omega^i$  we may assume  $i = n$ . Let  $N := \lambda_n = \mu_n$ . Then, by applying  $\omega^{(n+1)N}$  we may assume  $N = 0$ . Suppose now that  $\lambda, \mu$  is a counterexample. Then, by applying affine reflections in the first  $n-1$  coordinates only and by using (3.9) we may assume that  $\mu$  is in the fundamental alcove, i.e.,

$$(9.10) \quad \mu = (\underbrace{x+1, \dots, x+1}_{a \text{ times}}, \underbrace{x, \dots, x}_{n-a \text{ times}}, 0) \quad \text{with } 0 \leq a < n.$$

We necessarily have  $d := |\lambda| = |\mu| = xn + a$ . We proceed by induction on  $|d|$ , the case  $d = 0$  being trivial. Assume first that  $x \geq 0$ . Then there is  $j \leq n$  with  $\lambda_j > 0$ . After

applying the affine reflection  $s_\alpha$  where

$$(9.11) \quad \alpha := \begin{cases} \varepsilon_1 - \varepsilon_j & \text{if } j \leq a \text{ or } a = 0 \\ -\varepsilon_1 + \varepsilon_j + 1 & \text{otherwise} \end{cases}$$

to  $\lambda$  and  $\mu$  we may assume  $\lambda_1 > 0$ . That way, we have

$$(9.12) \quad \lambda \leq \mu \Leftrightarrow \omega(\lambda) \leq \omega(\mu) \Leftrightarrow s_n \omega(\lambda) \leq s_n \omega(\mu) \stackrel{(*)}{\Leftrightarrow} \omega(\lambda') \leq \omega(\mu') \Leftrightarrow \lambda' \leq \mu'$$

where  $(*)$  is the induction hypothesis.

For  $x < 0$  we proceed similarly. In that case, there is  $j \leq n$  with  $\lambda_j < 0$ . Then we use the affine reflection  $s_\alpha$  with

$$(9.13) \quad \alpha := \begin{cases} -\varepsilon_j + \varepsilon_n + 1 & \text{if } j \leq a \\ \varepsilon_j - \varepsilon_n & \text{otherwise} \end{cases}$$

to obtain  $\lambda_n < 0$ . Finally, we have

$$(9.14) \quad \lambda \leq \mu \Leftrightarrow s_n(\lambda) \leq s_n(\mu) \Leftrightarrow \omega^{-1} s_n(\lambda) \leq \omega^{-1} s_n(\mu) \stackrel{(*)}{\Leftrightarrow} \omega^{-1}(\lambda') \leq \omega^{-1}(\mu') \Leftrightarrow \lambda' \leq \mu'$$

□

**9.5. Corollary.** *There is a unique order relation on  $\Lambda$  whose restriction to each  $\Lambda_n$  is the Bruhat order.*

**9.6. Proposition.** *Let  $\lambda, \mu \in \Lambda$  with  $\lambda \preceq \mu$ . Then  $l(\lambda) \geq l(\mu)$ .*

*Proof:* Let  $\lambda, \mu$  be a counterexample. By Lemma 9.4 we may assume  $\lambda_n = 0$  and  $\mu_n > 0$ . Then  $(\omega^*)^{-1}(\mu) = (\mu_n - 1, \mu_1, \dots, \mu_{n-1}) \in \Lambda$  and  $(\omega^*)^{-1}(\lambda) \preceq (\omega^*)^{-1}(\mu)$ . Hence  $(\omega^*)^{-1}(\lambda) \in \Lambda$  by Lemma 7.3, i.e.,  $\lambda_n > 0$ . □

**9.7. Corollary.** *For every  $\lambda \in \Lambda$  there are only finitely many  $\mu \in \Lambda$  with  $\lambda \preceq \mu$ . In particular, the Bruhat order on  $\Lambda$  satisfies the ascending chain condition.*

*Proof:* Indeed,  $\lambda \preceq \mu$  implies that length and degree of  $\mu$  is bounded. □

**9.8. Corollary.** *For the Kazhdan-Lusztig involution holds  $d_{n-1}\pi_n = \pi_n d_n$ . In particular, there is an involution  $d$  of  $\mathcal{M}_\infty^{\text{pol}}$  which is compatible with all  $d_n$ .*

*Proof:* Proposition 9.6 implies that  $d_n$  preserves the kernel of  $\pi_n$ . Hence it induces a unique involution  $\tilde{d}$  of  $\mathcal{M}_{n-1}^{\text{pol}}$  with  $\tilde{d}\pi_n = \pi_n d_n$ . To show  $\tilde{d} = d_{n-1}$  it suffices to show  $\tilde{d}(M_0) = M_0$ ,  $\tilde{d}H_i = H_i^{-1}\tilde{d}$  for  $i = 1, \dots, n-2$  and  $\tilde{d}\omega_{n-1} = \omega_{n-1}\tilde{d}$ . The first statement is clear, the second follows from (9.2):

$$(9.15) \quad \tilde{d}H_i\pi_n = \tilde{d}\pi_n H_i = \pi_n d_n H_i = \pi_n H_i^{-1} d_n = H_i^{-1} \pi_n d_n = H_i^{-1} \tilde{d}\pi_n,$$

and the third from (9.4):

$$(9.16) \quad \tilde{d}\omega_{n-1}\pi_n = \tilde{d}\pi_n H_{n-1}\omega_n = \pi_n d_n H_{n-1}\omega_n = \pi_n H_{n-1}^{-1}\omega_n d_n = \omega_{n-1}\pi_n d_n = \omega_{n-1}\tilde{d}\pi_n.$$

□

Let  $\mathcal{M}_{++}^{\text{pol}}$  be the set of possibly infinite linear combinations  $\sum_{\lambda \in \Lambda} a_\lambda M^\lambda$  with  $a_\lambda \in v\mathbb{Z}[v]$ .

**9.9. Theorem.** *For every  $\lambda \in \Lambda$  there is a unique  $\underline{M}^\lambda \in \mathcal{M}_\infty^{\text{pol}}$  with  $d(\underline{M}^\lambda) = \underline{M}^\lambda$  and  $\underline{M}^\lambda \in M^\lambda + \mathcal{M}_{++}$ . This element is triangular with respect to the Bruhat order. Moreover,  $\underline{M}^\lambda = \lim_{n \rightarrow \infty} \underline{M}^{\lambda \leq n}$ .*

*Proof:* For any  $n \geq 2$  we have

$$(9.17) \quad \pi_n(\underline{M}^{\lambda \leq n}) = \begin{cases} \underline{M}^{\lambda < n} & \text{if } \lambda_n = 0 \\ 0 & \text{otherwise} \end{cases}$$

For  $\lambda_0 = 0$ , this follows from Corollary 9.8, otherwise it is implied by Proposition 9.6. This shows the existence of  $\underline{M}^\lambda$  and its triangularity (Corollary 9.5). For uniqueness, suppose there are two solutions  $\underline{M}_1$  and  $\underline{M}_2$ . Write  $m := \underline{M}_1 - \underline{M}_2 = \sum_{\lambda \in \Lambda} a_\lambda M^\lambda$  and let  $\mu$  be maximal with  $a_\mu \neq 0$  (see Corollary 9.7). Then  $d(a_\mu) = a_\mu$  and  $a_\mu \in v\mathbb{Z}[v]$  which is impossible. □

An analogous statement holds for Macdonald polynomials:

**9.10. Theorem.** *Let  $\lambda \in \Lambda$ . Then for any  $n \geq 2$  we have*

$$(9.18) \quad \pi_n(\mathcal{E}_{\lambda \leq n}) = \begin{cases} \mathcal{E}_{\lambda < n} & \text{if } \lambda_n = 0; \\ 0 & \text{otherwise.} \end{cases}$$

*In particular,  $\mathcal{E}_\lambda := \lim_{n \rightarrow \infty} \mathcal{E}_{\lambda \leq n}$  exists. Moreover, the recursion formula (8.7) is still valid.*

*Proof:* Apply  $\pi_n$  to both sides of (8.7). If  $\lambda_n > 0$ , then  $m = n$  and  $\Phi_m = \bar{\Phi}_m = \omega$ . Thus (9.18) follows from (9.3). Otherwise, we apply (9.4). □

For the Cherednik operators we have:

**9.11. Proposition.** *Let  $\xi_i^{(n)}$  be the Cherednik operator (6.4) in  $n$  variables. Then the following commutation rules hold:*

$$(9.19) \quad \pi_n H_{n-1} \tilde{\omega}_n = v^{-1} \tilde{\omega}_{n-1} \pi_n$$

$$(9.20) \quad \pi_n \xi_i^{(n)} = \xi_i^{(n-1)} \pi_n \quad \text{for } i = 1, \dots, n-1$$

*In particular, the limit operator  $\xi_i := \lim_{n \rightarrow \infty} \xi_i^{(n)}$  exists and*

$$(9.21) \quad \xi_i(\mathcal{E}_\lambda) = q^{\lambda_i} t^{1-w^\lambda(i)} \mathcal{E}_\lambda.$$

*Moreover, the  $\mathcal{E}_\lambda$  are, up to a scalar, the only joint eigenvectors in  $\mathcal{P}_\infty$ .*

*Proof:* By Corollary 9.2 we may think of  $\pi_n$  as projection  $\mathcal{L}[z_1, \dots, z_{n+1}] \rightarrow \mathcal{L}[z_1, \dots, z_n]$ . Then (7.22) shows  $\pi_n H_{n-1} = v^{-1} \pi_n s_{n-1}$ . A direct calculation using (7.23) shows (9.19). This and the definition (6.4) shows (9.20). Equation (9.21) follows readily from (6.7). Finally assume  $\mathcal{E}$  is another eigenvector. Let  $z^\lambda$  be a monomial occurring in  $\mathcal{E}$  for which  $\lambda$  is maximal with respect to the Bruhat order. The triangularity of  $\xi_i$  shows that  $\mathcal{E}$  corresponds to the same eigenvalue as  $\mathcal{E}_\lambda$ . For suitable  $a$ , the  $x^\lambda$ -term of  $\mathcal{E}' := \mathcal{E} - a\mathcal{E}_\lambda$  cancels out. If  $\mathcal{E}' \neq 0$  we could replace  $\mathcal{E}$  by  $\mathcal{E}'$  and obtain a contradiction.  $\square$

## 10. The almost symmetric submodule

The elements  $\underline{M}^\lambda$  cannot form a basis of  $\mathcal{M}_\infty^{\text{pol}}$  since that space is far too big. To pin down the span we introduce for any  $\lambda \in \Lambda$  the notation  $\lambda_{\leq m} := (\lambda_1, \dots, \lambda_m)$  and  $\lambda_{> m} := (\lambda_{m+1}, \lambda_{m+2}, \dots)$ . For fixed  $m \geq 0$  we define  $\mathcal{M}(m) \subseteq \mathcal{M}_\infty^{\text{pol}}$ ,  $\mathcal{P}(m) \subseteq \mathcal{P}_\infty$  as the space of *m-symmetric elements*, i.e., elements  $\xi$  with

$$(10.1) \quad H_i(\xi) = v^{-1} \xi \quad \text{for all } i > m.$$

For  $\xi = \sum_\lambda a_\lambda M^\lambda \in \mathcal{M}_\infty^{\text{pol}}$  this condition simply means

$$(10.2) \quad a_\lambda = v^{\ell(w^\lambda) - \ell(w^\mu)} a_\mu$$

whenever  $\lambda_{\leq m} = \mu_{\leq m}$  and  $\lambda_{> m}$  is a permutation of  $\mu_{> m}$ . For  $\xi \in \mathcal{P}_\infty$  it means even simpler that  $\xi$  is symmetric in the variables  $z_{m+1}, z_{m+2}, \dots$ . This follows from

$$(10.3) \quad H_i - v^{-1} = -\frac{v^{-1} z_i - v z_{i+1}}{z_i - z_{i+1}} (1 - s_i).$$

A basis of  $\mathcal{M}(m)$  can be constructed as follows. Let  $\Lambda(m)$  be the set of  $\lambda \in \Lambda$  such that  $\lambda_{>m}$  is a partition. Then, for  $\lambda \in \Lambda(m)$  we define

$$(10.4) \quad M^{\lambda|m} := \sum_{\lambda'} v^{\ell(w^{\lambda'})} M^{\lambda \leq_m \lambda'},$$

where  $\lambda'$  runs through all permutations of  $\lambda_{>m}$  and where  $\lambda \leq_m \lambda'$  denotes the concatenation  $(\lambda_1, \dots, \lambda_m, \lambda'_1, \lambda'_2, \dots)$ .

Clearly we have  $\mathcal{M}(0) \subseteq \mathcal{M}(1) \subseteq \dots$  and  $\mathcal{P}(0) \subseteq \mathcal{P}(1) \subseteq \dots$ . Their unions are denoted by  $\mathcal{M}^{\text{as}}$  and  $\mathcal{P}^{\text{as}}$ , respectively, and their elements are called “almost symmetric”. We still have an isomorphism

$$(10.5) \quad \Psi : \mathcal{P}^{\text{as}} \xrightarrow{\sim} \mathcal{M}^{\text{as}}.$$

Also  $\mathcal{M}^{\text{as}}$  possesses a nice basis. For  $\lambda \in \Lambda$  we define its *partition length*  $\text{pl}(\lambda)$  as the minimal number  $m \geq 0$  such that  $\lambda_{>m}$  is a partition. For example,

$$(10.6) \quad \lambda = (1, 2, 1, 0, 2, 1, 0, 0, \dots) \quad \text{has} \quad \text{pl} = 4.$$

Moreover,  $\text{pl}(\lambda) = 0$  if and only if  $\lambda$  itself is a partition. Now we simply define

$$(10.7) \quad M^{\lambda|} := M^{\lambda|\text{pl}(\lambda)}.$$

For example

$$(10.8) \quad M^{(0,2)|} = M^{(0,2)} + vM^{(0,0,2)} + v^2M^{(0,0,0,2)} + \dots$$

**10.1. Theorem.** *The elements  $M^{\lambda|}$ ,  $\lambda \in \Lambda$ , form an  $\mathcal{L}$ -basis of  $\mathcal{M}^{\text{as}}$ . Moreover, the  $M^{\lambda|}$  with  $\text{pl}(\lambda) \leq m$  span  $\mathcal{M}(m)$ .*

*Proof:* First we show that the  $M^{\lambda|}$  span  $\mathcal{M}^{\text{as}}$ . Clearly, the  $M^{\mu|m}$  with  $\mu \in \Lambda(m)$  form a basis of  $\mathcal{M}(m)$ . Since  $\mathcal{M}^{\text{as}}$  is the union of the  $\mathcal{M}(m)$  it suffices to show that  $M^{\mu|m}$  is in the span of the  $M^{\lambda|}$ . If  $m = \text{pl}(\mu)$ , then there is nothing to show. Thus assume  $m > \text{pl}(\mu)$ , i.e.,  $\mu'' := \mu_{\geq m} \in \Lambda(0)$ . For each part  $a$  of  $\mu''$  let  $\mu''_a$  be obtained from  $\mu''$  by putting  $a$  in front and omitting one occurrence of  $a$ . E.g. if  $\mu'' = (4, 3, 3, 0, \dots)$ , then

$$(10.9) \quad \mu''_4 = \mu'' = (4, 3, 3, 0, \dots), \quad \mu''_3 = (3, 4, 3, 0, \dots), \quad \mu''_0 = (0, 4, 3, 3, 0, \dots).$$

Assume  $a$  occurs in  $\mu''$  for the first time in position  $i_a$ . Put  $\mu' := \mu_{\leq m-1}$  and  $\mu_a := \mu' \mu''_a$ . Then we have the formula

$$(10.10) \quad M^{\mu|m-1} = \sum_a v^{i_a-1} M^{\mu_a|m} = M^{\mu|m} + \sum_{a \neq \mu_m} v^{i_a-1} M^{\mu_a|m}$$

which proves the claim by induction.

As for the linear independence, assume  $\sum_{\lambda} a_{\lambda} M^{\lambda} = 0$ . Let  $\lambda$  be maximal with respect to the lexicographic order with  $a_{\lambda} \neq 0$ . Then  $M^{\lambda}$  occurs only in  $M^{\lambda}$ , which yields the contradiction  $a_{\lambda} = 0$ .  $\square$

The main reason for introducing  $\mathcal{M}^{\text{as}}$  is the following

**10.2. Theorem.** *The elements  $\underline{M}^{\lambda}$ ,  $\lambda \in \Lambda$ , form an  $\mathcal{L}$ -basis of  $\mathcal{M}^{\text{as}}$ . Moreover, the  $\underline{M}^{\lambda}$  with  $\text{pl}(\lambda) \leq m$  span  $\mathcal{M}(m)$ .*

*Proof:* Lemma 5.4 implies  $\underline{M}^{\lambda} \in \mathcal{M}(m) \subseteq \mathcal{M}^{\text{as}}$  for  $m \geq \text{pl}(\lambda)$ . Now fix  $d \geq 0$  and  $m \geq 0$ . Let  $\lambda \in \Lambda(m)$  with  $|\lambda| = d$ . Then in the expansion  $\underline{M}^{\lambda} = \sum_{\mu} a_{\mu} M^{\mu}$  only those  $\mu$  occur with  $|\mu| = d$ ,  $\mu \in \Lambda(m)$  and  $\mu \preceq \lambda$ . Moreover,  $a_{\lambda\lambda} = 1$ . Thus the transition matrix  $(a_{\lambda\mu})$  is unitriangular and finite, hence invertible. This implies that every  $M^{\mu}$  is in the span of the  $\underline{M}^{\lambda}$ .  $\square$

For the Macdonald polynomials we have

**10.3. Lemma.** *The operators  $\Phi_m$ , and  $\bar{\Phi}_m$  act on  $\mathcal{P}^{\text{as}}(n)$  for any  $n \geq m$ . In particular,  $\mathcal{E}_{\lambda} \in \mathcal{P}_q(n) \subseteq \mathcal{P}_q^{\text{as}}$  for any  $n \geq l(\lambda)$ .*

*Proof:* This follows from the fact that  $\Phi_m$  and  $\bar{\Phi}_m$  commute with  $H_n$  for any  $n > m$ .  $\square$

Note however that the  $\mathcal{E}_{\lambda}$  do not span  $\mathcal{P}_q^{\text{as}}$ . For example we have

**10.4. Lemma.** *Let  $\mathcal{P}' \subseteq \mathcal{P}_q^{\text{as}}$  be the  $\mathcal{L}_q$ -span of the  $\mathcal{E}_{\lambda}$ ,  $\lambda \in \Lambda$ . Then  $\mathcal{P}_q(0) \cap \mathcal{P}' = \mathcal{L}_q$ .*

*Proof:* Let  $\mathcal{E} = \sum_{\lambda} c_{\lambda} \mathcal{E}_{\lambda}$  be a finite linear combination which is not constant. Choose  $\lambda \in \Lambda$  with  $c_{\lambda}$  such that  $m := l(\lambda)$  is maximal. Then  $m \geq 1$ . It is well known (see, e.g., [Kn1] Thm. 4.2 or Lemma 11.5 below) that  $H_m(\mathcal{E}_{\lambda}) = a\mathcal{E}_{s_m(\lambda)} + b\mathcal{E}_{\lambda}$  with  $a \neq 0$ . Thus,  $\mathcal{E}$  cannot be symmetric.  $\square$

## 11. The scalar product and composition Kostka functions

Recall the following notation from Macdonald's book [M2] III.2: for any integer  $m \geq 0$  put  $\varphi_m(t) := \prod_{i=1}^m (1 - t^i)$ . For a partition  $\lambda \in \Lambda(0)$  and an integer  $a \geq 0$  let  $m_a(\lambda) := \#\{i \geq 1 \mid \lambda_i = a\}$  and

$$(11.1) \quad b_{\lambda}(t) := \prod_{a \geq 1} \varphi_{m_a(\lambda)}(t)$$

**11.1. Theorem.** *We equip  $\mathbb{Q}(v)$  with the  $v$ -adic topology. Then, there is a unique  $\mathcal{L}$ -linear continuous scalar product  $\mathcal{M}^{\text{as}} \times \mathcal{M}^{\text{as}} \rightarrow \mathbb{Q}(v)$  such that the  $M^{\lambda}$ ,  $\lambda \in \Lambda$ , are orthonormal.*

It has the property that

$$(11.2) \quad \langle M^{\lambda|m}, M^{\mu|m} \rangle = \frac{\delta_{\lambda\mu}}{b_{\lambda_{>m}}(t)} \quad \text{for all } m \geq 0 \text{ and } \lambda, \mu \in \Lambda(m).$$

Moreover, the operators  $H_w$ ,  $w \in S_\infty$ , are selfadjoint.

*Proof:* Uniqueness is clear since the  $M^\lambda$  are dense in  $\mathcal{M}^{\text{as}}$ . In view of Theorem 10.1, for existence it suffices to show (11.2). If  $\lambda \neq \mu$ , then  $\lambda_{\leq m} \lambda' \neq \mu_{\leq m} \mu'$  where  $\lambda', \mu'$  are permutations of  $\lambda_{>m}, \mu_{>m}$ , respectively. This shows  $\langle M^{\lambda|m}, M^{\mu|m} \rangle = 0$  for  $\lambda \neq \mu$  and it remains to compute  $\langle M^{\lambda|m}, M^{\lambda|m} \rangle$ . For this, we may clearly assume  $m = 0$ . Then

$$(11.3) \quad A_\infty := \langle M^{\lambda|0}, M^{\lambda|0} \rangle = \sum_{\lambda' \in S_\infty \lambda} v^{2\ell(w^{\lambda'})}.$$

For  $n \geq l(\lambda)$  we let  $A_n$  be the subsum with  $\lambda' \in S_n \lambda$ . Let  $S_\lambda$  be the isotropy group of  $\lambda$  in  $S_n$ . For a finite Coxeter group  $H$  let  $p_H(t)$  be the function  $\sum_{w \in H} t^{\ell(w)}$ . Then

$$(11.4) \quad A_n = \frac{p_{S_n}(t)}{p_{S_\lambda}(t)}.$$

We have

$$(11.5) \quad p_{S_n}(t) = (1+t)(1+t+t^2) \dots (1+t+\dots+t^{n-1}) = \frac{\prod_{i=1}^n (1-t^i)}{(1-t)^n}.$$

Put  $m_a := m_a(\lambda)$ . From  $S_\lambda = S_{m_0} \times S_{m_1} \times \dots$  with  $m_0 + m_1 + \dots = n$  we get

$$(11.6) \quad p_{S_\lambda}(t) = \frac{\prod_{i=1}^{m_0} (1-t^i)}{(1-t)^{m_0}} \frac{\prod_{i=1}^{m_1} (1-t^i)}{(1-t)^{m_1}} \frac{\prod_{i=1}^{m_2} (1-t^i)}{(1-t)^{m_2}} \dots = \frac{\prod_{i=1}^{m_0} (1-t^i)}{(1-t)^n} b_\lambda(t)$$

Hence

$$(11.7) \quad A_n = \frac{\prod_{i=m_0+1}^n (1-t^i)}{b_\lambda(t)} \xrightarrow{n \rightarrow \infty} \frac{1}{b_\lambda(t)} = A_\infty.$$

Finally, formula (3.4) shows that the matrix of  $H_i$  with respect to the basis  $M^\lambda$  is symmetric. This implies that all operators  $H_w$ ,  $w \in S_\infty$ , are selfadjoint.  $\square$

At last, we link Kazhdan-Lusztig polynomials and Macdonald polynomials in the following

**Definition:** For  $\lambda, \mu \in \Lambda$  we define the *composition Kostka function* as

$$(11.8) \quad K_{\lambda\mu}(q, t) := \langle \underline{M}^\lambda, \Psi(\tilde{\mathcal{E}}_\mu) \rangle.$$

In [M1], Macdonald constructed a two-parameter function  $K_{\lambda\mu}(q, t)$  where  $\lambda$  and  $\mu$  are partitions and conjectured that they are polynomials in  $q$  and  $t$  with non-negative integers

as coefficients. The fact, that  $K_{\lambda\mu}(q, t)$  is a polynomial was proved almost simultaneously in [GR], [GT], [Ki], [Kn1], and [Sa]. The remaining positivity conjecture was finally settled affirmatively by Haiman [Ha]. We are going to show (Theorem 11.8) that our  $K_{\lambda\mu}$  coincide with Macdonald's in case  $\lambda, \mu$  are partitions. The main “result” of this paper is the following

**11.2. Conjecture.** *For all  $\lambda, \mu \in \Lambda$  holds  $K_{\lambda\mu}(q, t) \in \mathbb{N}[v, q]$ .*

As for the evidence, we have

- The conjecture is true for  $q = 0$ . In fact,

**11.3. Lemma.**  *$K_{\lambda\mu}(0, t)$  is a Kazhdan-Lusztig polynomial.*

*Proof:* Given  $\mu \in \Lambda$ , the expansion of the recursion formula (8.12) for  $q = 0$  gives

$$(11.9) \quad \Psi(\tilde{\mathcal{E}}_\mu)|_{q=0} = H_{c_{m_d}} \dots H_{c_{m_2}} H_{c_{m_1}}(M_0)$$

with uniquely determined numbers  $m_d \geq \dots \geq m_2 \geq m_1 \geq 1$ . It is easy to see that  $c_{m_d} c_{m_{d-1}} \dots c_{m_1}$  is a reduced decomposition of  $m_{-\tau}$ . This implies

$$(11.10) \quad \Psi(\tilde{\mathcal{E}}_\mu)|_{q=0} = M^\mu$$

and therefore

$$(11.11) \quad \underline{M}^\lambda = \sum_{\mu} K_{\lambda\mu}(0, t) M^\mu.$$

□

**Remark:** After the first release of this paper in the arxiv the specialization statement (11.10) has been generalized to arbitrary root systems by Ion, [Ion]. This led him to speculations about Macdonald positivity for arbitrary root systems. As explained in the introduction, such a thing does not even exist for the root system  $A_{n-1}$  where  $n$  is fixed. More precisely, any generalization of Macdonald positivity to arbitrary root systems would require completely new ideas if it exists at all.

- We can almost prove polynomiality. This is our “real” main result.

**11.4. Theorem.** *For all  $\lambda, \mu \in \Lambda$  holds  $K_{\lambda\mu}(q, t) \in \mathbb{Z}[v, v^{-1}, q]$ .*

*Proof:* For  $m \geq 0$  and  $\lambda \in \Lambda(m)$  define  $\tilde{M}^{\lambda|m} := b_{\lambda_{>m}}(v^2) M^{\lambda|m}$ . If  $l(\mu) \leq m$ , then we can expand

$$(11.12) \quad \tilde{\mathcal{E}}_\mu = \sum_{\tau \in \Lambda(m)} c_{\mu\tau} \tilde{M}^{\tau|m}.$$



It has been shown in [Kn1] Thm. 5.4 that the coefficients  $c_{\mu\tau}$  are in  $\mathbb{Z}[q, v, v^{-1}]$ . If  $\lambda$  has  $\text{pl}(\lambda) \leq m$ , then  $\underline{M}^\lambda$  has an expansion in terms of the  $M^{\tau|m}$  with polynomial coefficients (Theorem 10.1). The claim follows from (11.2).  $\square$

**Remark:** The non-appearance of negative powers of  $v$  is equivalent to Conjecture 8.4. In any case, by tracing through all definitions, it would be possible to give an explicit upper bound for the pole order of  $K_{\lambda\mu}(q, t)$  at  $v = 0$  depending only on  $\mu$ .

- Using a computer, we tested the conjecture in thousands of cases.
- Finally, as mentioned, the conjecture holds for  $\lambda, \mu \in \Lambda(0)$  since Macdonald's Kostka functions are special cases of ours. We start with a lemma.

**11.5. Lemma.** *For  $i \geq 1$ ,  $\mu \in \Lambda$  with  $s_i(\mu) \neq \mu$  let*

$$(11.13) \quad f_\mu := q^{\mu_i - \mu_{i+1}} t^{w^\mu(i+1) - w^\mu(i)}, \quad A_\mu := \frac{v - v^{-1}f_\mu}{1 - f_\mu}.$$

*Then*

$$(11.14) \quad (H_i - v^{-1})(\mathcal{E}_\mu) = A_\mu(\mathcal{E}_{s_i(\mu)} - \mathcal{E}_\mu).$$

*Proof:* From [Kn1] Theorem 4.2 one deduces the formula

$$(11.15) \quad (H_i - v^{-1})E_{s_i(\mu)} = v^{-1}E_\mu - \frac{v^{-1} - vf_\mu}{1 - f_\mu}E_{s_i(\mu)}$$

under the provision  $\mu_i > \mu_{i+1}$ . Here  $E_\mu$  is the Macdonald polynomial with the  $z^\mu$ -coefficient normalized to 1. Now  $f_{s_i(\mu)} = f_\mu^{-1}$ . Thus replacing  $\mu$  by  $s_i(\mu)$  in (11.15) results in

$$(11.16) \quad (H_i - v^{-1})E_\mu = v^{-1}E_{s_i(\mu)} - A_\mu E_{s_i(\mu)}.$$

Let  $c_\lambda$  be the normalization factor (6.11). Then formula (11.14) amounts to  $v^{-1}c_\mu/c_{s_i(\mu)} = A_\mu$ . This is readily verified using the fact that  $c_\mu$  and  $c_{s_i(\mu)}$  differ in only one factor namely the contribution of the box  $(i+1, \mu_i+1)$  and  $(i, \mu_i+1)$ , respectively. This proves (11.14) in the case  $\mu_i < \mu_{i+1}$ . The other case can be easily deduced from that using the Hecke relation (2.13).  $\square$

**11.6. Corollary.** *Let  $\lambda, \mu \in \Lambda$ ,  $i \geq 1$  with  $\lambda_i \geq \lambda_{i+1}$  and  $\mu_i > \mu_{i+1}$ . Then*

$$(11.17) \quad K_{\lambda s_i(\mu)} = vK_{\lambda\mu}.$$

*Proof:* Let  $\Xi := H_i - v^{-1}$ . Then  $\Xi(\underline{M}^\lambda) = 0$  (Lemma 5.4) and  $\Xi(\tilde{\mathcal{E}}_\mu) = v^{-1}A_\mu(\tilde{\mathcal{E}}_{s_i(\mu)} - v\tilde{\mathcal{E}}_\mu)$  (Lemma 11.5). Moreover,  $\Xi$  is selfadjoint. Hence

$$(11.18) \quad 0 = \langle \Xi(\underline{M}^\lambda), \Psi(\tilde{\mathcal{E}}_\mu) \rangle = v^{-1}A_\mu \langle \underline{M}^\lambda, \Psi(\tilde{\mathcal{E}}_{s_i(\mu)} - v\tilde{\mathcal{E}}_\mu) \rangle = v^{-1}A_\mu(K_{\lambda s_i(\mu)} - vK_{\lambda\mu}).$$

$\square$

This result reduces the computation of  $K_{\lambda\mu}$  to the case where  $\mu_i \geq \mu_{i+1}$  whenever  $\lambda_i \geq \lambda_{i+1}$ . In particular, if  $\lambda$  is a partition, one may assume that  $\mu$  is a partition, as well.

Now we introduce the symmetric (i.e., original) Macdonald functions. Let  $\lambda \in \Lambda(0)$  be a partition. The subspace of  $\mathcal{L}_q[z_1, \dots, z_n]$  spanned by the  $\mathcal{E}_\mu$ ,  $\mu \in S_n \lambda_{\leq n}$  contains a unique symmetric polynomial  $\mathcal{J}_{\lambda_{\leq n}}$  whose  $z^\lambda$ -coefficient is

$$(11.19) \quad \prod_{s \in \lambda} \left(1 - q^{a_\lambda(s)} t^{l_\lambda(s)+1}\right).$$

(note the small difference to (6.11)). It follows from (9.18) that the  $\mathcal{J}_{\lambda_{\leq n}}$  are compatible and therefore have a limit  $\mathcal{J}_\lambda \in \mathcal{P}_q(0)$ , the *symmetric Macdonald function*.

**11.7. Lemma.** *For  $\lambda, \mu \in \Lambda(0)$  holds  $\langle \underline{M}^\lambda, \Psi(\mathcal{J}_\mu) \rangle = K_{\lambda\mu}$ .*

*Proof:* We work first with a finite number of  $n$  variables. Let  $\lambda, \mu \in \Lambda_n$  be partitions. It was shown in the proof of [Kn1] Thm. 6.1 that

$$(11.20) \quad \mathcal{J}_\mu = c_\mu \sum_{w \in W_f} v^{\ell(w)} H_w^{-1}(\mathcal{E}_{\mu^-}) \quad \text{where}$$

$$(11.21) \quad \mu^- := w_0(\mu), \text{ and } c_\mu := \frac{(1-t)^n}{\prod_{i=1}^m (1 - q^{\mu_i} t^{n-m+i}) \prod_{a=1}^{n-m} (1 - t^a)} \text{ with } m := l(\mu).$$

Thus,

$$(11.22) \quad \begin{aligned} \langle \underline{M}^\lambda, \Psi(\mathcal{J}_\mu) \rangle &= c_\mu \sum_{w \in W_f} v^{\ell(w)} \langle H_w^{-1} \underline{M}^\lambda, \Psi(\mathcal{E}_{\mu^-}) \rangle = \\ &= d_\mu \langle \underline{M}^\lambda, \Psi(\mathcal{E}_{\mu^-}) \rangle = d_\mu \langle \underline{M}^\lambda, \Psi(\mathcal{E}_\mu) \rangle \end{aligned}$$

where the second equality is Lemma 5.4 and the last is Corollary 11.6. The coefficient is

$$(11.23) \quad d_\mu = c_\mu \sum_{w \in W_f} t^{\ell(w)} = c_\mu \frac{\prod_{a=1}^n (1 - t^a)}{(1-t)^n} = \prod_{i=1}^m \frac{1 - t^{n-m+i}}{1 - q^{\lambda_i} t^{n-m+i}}.$$

The assertion follows from  $\lim_{n \rightarrow \infty} d_\mu = 1$ . □

**11.8. Theorem.** *Assume that both  $\lambda$  and  $\mu$  are partitions. Then  $K_{\lambda\mu}$  coincides with Macdonald's  $q, t$ -Kostka function.*

*Proof:* On one hand,  $\mathcal{J}_\mu|_{q=0}$  equals  $\tilde{M}^{\mu|0}$  ([Kn1] Thm. 6.2). On the other hand, it is also the Hall-Littlewood polynomial  $Q_\mu$  ([M2] VI (8.4)ii). Let  $\langle \cdot, \cdot \rangle_{\text{HL}}$  be the scalar product of [M2] III.4 on symmetric functions making the Hall-Littlewood functions orthogonal. Then the comparison of (11.2) with the scalar product of Hall-Littlewood functions shows

$$(11.24) \quad \langle \Psi(f), \Psi(g) \rangle = \langle f, g \rangle_{\text{HL}}$$

for any two symmetric functions  $f, g \in \mathcal{P}(0)$ . Since  $\lambda \in \Lambda(0)$  we have  $\underline{M}^\lambda = \Psi(s_\lambda)$  (Theorem 5.5). Thus we have

$$(11.25) \quad K_{\lambda\mu} = \langle \underline{M}^\lambda, \Psi(\mathcal{J}_\mu) \rangle = \langle s_\lambda, \mathcal{J}_\mu \rangle_{\text{HL}}.$$

The last expression is just Macdonald's definition of  $K_{\lambda\mu}$ . □

## 12. A refinement

The recursive formula (8.12) can be expanded to give a closed formula for the polynomials  $\tilde{\mathcal{E}}_\lambda$ . For this we define the *column-length* of  $s \in \lambda$  as

$$(12.1) \quad c_\lambda(s) := \#\{k < i \mid j \leq \lambda_k + 1\} + \#\{k \geq i \mid j \leq \lambda_k\}.$$

If  $\lambda$  is a partition, then  $c_\lambda(s)$  is the length of the column containing  $s$ . Now we enumerate the boxes of  $\lambda$  from the top to the bottom starting with the *rightmost* column and working to the left. For example  $\lambda = (3, 0, 1, 2, 0, \dots)$  gives

$$(12.2) \quad \begin{array}{|c|c|c|} \hline 4 & 2 & 1 \\ \hline & & \\ \hline & 5 & \\ \hline 6 & 3 & \\ \hline \end{array}$$

For  $i = 1, \dots, |\lambda|$  we put  $c_i := c_\lambda(s_i)$ . In the example above we get the sequence

$$(12.3) \quad 1, 2, 3, 3, 4, 4$$

For  $m \geq 1$  we define the operators

$$(12.4) \quad X_m^{(0)} := \Phi_m, \quad X_m^{(1)} := -\bar{\Phi}_m.$$

A *marked* diagram  $\lambda$  is diagram  $\lambda$  together with a subset  $S \subseteq \lambda$  of boxes. Let  $\varepsilon_i = 1$  if  $s_i \in S$  and  $\varepsilon_i = 0$  otherwise. Then for a marked diagram  $S \subseteq \lambda$  (with  $n := |\lambda|$ ) we define the partial Macdonald polynomial as

$$(12.5) \quad \tilde{\mathcal{E}}_\lambda := X_{c_n}^{(\varepsilon_n)} \dots X_{c_1}^{(\varepsilon_1)}(1).$$

For example, the marked diagram of the shape (12.2)

$$(12.6) \quad \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline & & \\ \hline \blacksquare & & \\ \hline \blacksquare & \blacksquare & \\ \hline \end{array}$$

gives

$$(12.7) \quad \tilde{\mathcal{E}}_\lambda = X_4^{(0)} X_4^{(1)} X_3^{(0)} X_3^{(1)} X_2^{(1)} X_1^{(0)}(1) = -\Phi_4 \bar{\Phi}_4 \Phi_3 \bar{\Phi}_3 \bar{\Phi}_2 \Phi_1(1).$$

**12.1. Theorem.** *The Macdonald polynomial  $\tilde{\mathcal{E}}_\lambda$  can be expressed as*

$$(12.8) \quad \tilde{\mathcal{E}}_\lambda = \sum_S q^{A_\lambda} t^{L_\lambda} \tilde{\mathcal{E}}_\lambda$$

where  $S$  runs through all markings of  $\lambda$  and where

$$(12.9) \quad A_\lambda := \sum_{s \in S} (a_\lambda(s) + 1), \quad L_\lambda := \sum_{s \in S} (l_\lambda(s) + 1).$$

*Proof:* For a non-empty diagram  $\mu$  we define the following operation: take the last row, remove its leftmost box and put the remainder of the row on top, e.g.,

$$(12.10) \quad \begin{array}{|c|c|c|} \hline 4 & 2 & 1 \\ \hline & & \\ \hline 5 & & \\ \hline 6 & 3 & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 3 & & \\ \hline 4 & 2 & 1 \\ \hline & & \\ \hline 5 & & \\ \hline \end{array}$$

The result is  $\lambda^*$ . Moreover, it is easily verified that the number, the arm-length, the leg-length, and the column-length of the surviving boxes don't change. Let  $s \in \lambda$  be the bottom left box. Then  $c_\lambda(s) = l(\lambda) = m$ ,  $a_\lambda(s) + 1 = \lambda_m$ , and  $l_\lambda(s) + 1 = a$  (defined in (8.5)). Thus, (12.8) is an expansion of (8.12).  $\square$

Accordingly, if we define the marked composition Kostka function as

$$(12.11) \quad K_{\lambda\mu}(t) := t^{L_\mu} \langle \underline{M}^\lambda, \Psi(\tilde{\mathcal{E}}_\mu), \rangle,$$

then we have

$$(12.12) \quad K_{\lambda\mu}(q, t) = \sum_S q^{A_\mu} K_{\lambda\mu}(t).$$

The same proof as for  $K_{\lambda\mu}$  shows  $K_{\lambda\mu} \in \mathbb{Z}[v, v^{-1}]$ . Indeed, the following seems to be true:

**12.2. Conjecture.** *For all  $\lambda \in \Lambda$  and all marked diagrams  $\mu$  holds  $K_{\lambda\mu} \in \mathbb{N}[v]$ .*

Observe that this conjecture is indeed stronger than Conjecture 11.2 since there are plenty of marked diagram with the same exponent  $A_\mu$ . One of the simplest examples is  $\lambda = (3, 1)$  and  $\mu = (2, 2)$ . Here

$$(12.13) \quad K_{\lambda\mu} = t + tq + t^2q$$

and all summands come from different marked diagrams namely

$$(12.14) \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \square & \blacksquare \\ \hline \square & \square \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \blacksquare \\ \hline \end{array}$$

Another example is  $\lambda = (3, 1, 1)$ ,  $\mu = (2, 2, 1)$ . Here

$$(12.15) \quad K_{\lambda\mu} = t + (t^2 + t^3)q + (t + t^2)q + t^3q^2$$

where the summands come from

$$(12.16) \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \square & \blacksquare \\ \hline \square & \square \\ \hline \square & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \blacksquare \\ \hline \square & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \square & \blacksquare \\ \hline \square & \blacksquare \\ \hline \square & \blacksquare \\ \hline \end{array}$$

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